

# A TUTTE-TYPE CHARACTERIZATION FOR GRAPH FACTORS

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ABSTRACT. Let  $G$  be a connected general graph. Let  $f: V(G) \rightarrow \mathbb{Z}^+$  be a function. We show that  $G$  satisfies the Tutte-type condition

$$o(G - S) \leq f(S) \quad \text{for all vertex subsets } S,$$

if and only if it contains a colored  $J_f^*$ -factor for any 2-end-coloring, where  $J_f^*(v)$  is the union of all odd integers smaller than  $f(v)$  and the integer  $f(v)$  itself. This is a generalization of the  $(1, f)$ -odd factor characterization theorem, and answers a problem of Cui and Kano. We also derive an analogous characterization for graphs of odd orders, which addresses a problem of Akiyama and Kano.

## 1. INTRODUCTION

This paper concerns Tutte type conditions and the existence of factors in general graphs. A considerable large number of literatures on graph factors can be found from Akiyama and Kano's book [1], and from Liu and Yu's book [12].

Tutte's theorem [11] states that a graph  $G$  has a perfect matching if and only if

$$(1.1) \quad o(G - S) \leq |S| \quad \text{for any vertex subset } S,$$

where  $o(G - S)$  denotes the number of odd components of the subgraph  $G - S$ . Let  $H: V(G) \rightarrow 2^{\mathbb{N}}$  be a set-valued function. A spanning subgraph  $F$  of  $G$  is said to be an  $H$ -factor if  $\deg_F(v) \in H(v)$ . Let  $f: V(G) \rightarrow \mathbb{Z}^+$  be an odd-integer-valued function. The factor  $F$  is said to be a  $(1, f)$ -odd factor if it is an  $H$ -factor where  $H(v) = \{1, 3, 5, \dots, f(v)\}$ . In particular, a perfect matching is called a 1-factor.

Lovász [6] proposed the *degree prescribed subgraph problem* of determining the distance of a factor from a given integer set function. He [7] considered it with the restriction that the given set function  $H$  is allowed, i.e., that every gap of the set  $H(v)$  for each vertex  $v$  is at most two. He also showed that the problem is NP-complete when the function  $H$  is not allowed. Cornuéjols [3] provided a polynomial Edmonds-Johnson type alternating forest algorithm for the degree prescribed subgraph problem with  $H$  allowed, which implies a Gallai-Edmonds type structure theorem.

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For convenience, we denote

$$J_n = \begin{cases} \{1, 3, 5, \dots, n\}, & \text{if } n \text{ is odd;} \\ \{1, 3, 5, \dots, n-1, n\}, & \text{if } n \text{ is even.} \end{cases}$$

Define  $J_f(v) = J_{f(v)}$  for all vertices  $v$ . Under this notation,  $J_f$ -factors are exactly  $(1, f)$ -odd factors when  $f(v)$  is odd for each vertex  $v$ . Amahashi [2] gave a Tutte-type characterization for graphs having a “global odd factor”.

**Theorem 1.1** (Amahashi). *Let  $n \geq 2$  be an integer. A general graph  $G$  has a  $J_{2n-1}$ -factor if and only if*

$$(1.2) \quad o(G - S) \leq (2n - 1)|S| \quad \text{for all vertex subsets } S.$$

By “global odd factor” we mean that the coefficient  $(2n - 1)$  in Condition (1.2) is an odd integer independent of the vertices. By generalizing the constant  $2n - 1$  to an odd-valued function  $f(v)$ , Cui and Kano [4] obtained Theorem 1.1 as follows. We denote  $f(S) = \sum_{v \in S} f(v)$  for any vertex subset  $S$ .

**Theorem 1.2** (Cui and Kano). *Let  $G$  be a general graph of even order. Let  $f: V(G) \rightarrow \mathbb{Z}^+$  be a function such that  $f(v)$  is odd for each vertex  $v$ . The graph  $G$  has a  $J_f$ -factor if and only if*

$$(1.3) \quad o(G - S) \leq f(S) \quad \text{for all vertex subsets } S.$$

They [4] also proposed the characterization problem for the existence of a “global even factor”.

**Problem 1.3** (Cui and Kano). *Is it possible to characterize graphs that satisfy the condition*

$$(1.4) \quad o(G - S) \leq 2n|S| \quad \text{for all vertex subsets } S,$$

*in terms of factors?*

In analog with Theorem 1.1, the present authors [8] obtained the following partial answer to Problem 1.3.

**Theorem 1.4** (Lu and Wang). *Let  $n \geq 2$ . Let  $G$  be a simple connected graph satisfying Condition (1.4). Then  $G$  contains a  $J_{2n}$ -factor.*

In the spirit of Cui and Kano’s generalizing Theorem 1.1, Egawa Kano, and Yan [5] generalized Theorem 1.4 by allowing the constant  $2n$  to vary as a function.

**Theorem 1.5** (Egawa et al.). *Let  $G$  be a simple connected graph. Let  $f: V(G) \rightarrow \mathbb{Z}^+$  be a function. If  $G$  satisfies Condition (1.3), then  $G$  has a  $J_f$ -factor.*

By setting  $S = \emptyset$ , we see that Problem 1.3 involves only graphs of even order. Taking account of graphs of odd order, Akiyama and Kano [1, Problem 6.14 (2)] presented the following problem in the same manner.

**Problem 1.6** (Akiyama and Kano). *Let  $G$  be a connected simple graph. Let  $f: V(G) \rightarrow \mathbb{Z}^+$  be a function such that  $f(v)$  is even for each vertex  $v$ . If*

$$(1.5) \quad o(G - S) \leq f(S) \quad \text{for all nonempty vertex subsets } S,$$

*then what factor or property does  $G$  have?*

We will characterize graphs of both even and odd orders satisfying the aforementioned Tutte-type conditions, in terms of the so-called colored factors. In fact, characterization results for the degree prescribed subgraph problem are rather scanty. Examples include Theorem 1.2, and those on parity interval factors [7], on the general antifactor problem [9], and on prescriptions whose gaps have the same parity [10]. In this paper, we provide one more member to the family of characterization results on graph factors, i.e., a graph of even order satisfies the Tutte-type condition (1.3) if and only if it contains a colored  $J_f^*$ -factor for any 2-end-coloring; see Theorem 3.1. It reduces to Theorem 1.2 by restricting the function  $f$  to be odd-integer-valued, and to Theorem 1.5 by coloring all ends in red. It is also an answer to Problem 1.3. Together with Theorem 3.2, which deals with graphs of odd orders, we obtain an answer to Problem 1.6.

## 2. PRELIMINARY

Let  $G$  be a general graph allowing both loops and parallel edges, with vertex set  $V(G)$  and edge set  $E(G)$ . The idea of coloring each end of every edge is due to Lovász [7]. In this section, we give an overview of his idea that help interprets negative degrees of vertices, based on which his structural description works for general graphs.

**2.1. Interpreting negative degrees of vertices.** We say that a general graph is *2-end-colored* (or with a *2-end-coloring*) if every end of every its edge is colored in red or in green, and if the loop ends receive the same color for each loop. We call a loop with two red ends a red loop, and a loop with two green ends a green loop. Let  $G$  be a general graph with a 2-end-coloring. One associates every edge  $e$  a characteristic function  $e: V(G) \rightarrow \{0, \pm 1, \pm 2\}$ , defined by

$$(2.1) \quad e(v) = \begin{cases} 2, & \text{if } e \text{ is a red loop and } v \text{ is the center of } e; \\ -2, & \text{if } e \text{ is a green loop and } v \text{ is the center of } e; \\ 1, & \text{if } e \text{ is not a loop and } v \text{ is a red end of } e; \\ -1, & \text{if } e \text{ is not a loop and } v \text{ is a green end of } e; \\ 0, & \text{otherwise, i.e., if } v \text{ is not incident with } e. \end{cases}$$

Define the *colored degree* of a vertex  $v$  by

$$\Phi_G(v) = \sum_{e \in E(G)} e(v).$$

Alternatively, one may consider every red end is weighted by  $+1$ , and every green end is weighted by  $-1$ . In this way, the colored degree of a vertex  $v$  is the weight sum of ends that incident with  $v$ .

**2.2. The structural description.** A set  $\{h_1, h_2, \dots, h_m\}$  of increasing integers is said to be *allowed* if  $h_{i+1} - h_i \leq 2$  for all  $1 \leq i \leq m-1$ . Let  $H: V(G) \rightarrow 2^{\mathbb{Z}}$  be an allowed set function, that is, the set  $H(v)$  is allowed for each vertex  $v$ . A factor  $F$  of the graph  $G$  is said to be a *colored  $H$ -factor* if  $\Phi_F(v) \in H(v)$  for each vertex  $v$ . The distance of the colored degree  $\Phi_F(v)$  from the set  $H(v)$  is defined by

$$\text{dist}_F(v, H(v)) = \min\{|\Phi_F(v) - h| : h \in H(v)\}.$$

Lovász [7] introduced the functions

$$(2.2) \quad \delta_H(F) = \sum_{v \in V(G)} \text{dist}_F(v, H(v)), \quad \text{and}$$

$$(2.3) \quad \delta(H) = \min\{\delta_H(F) : F \text{ is a subgraph of } G\}.$$

The factor  $F$  is said to be  *$H$ -optimal* if  $\delta_H(F) = \delta(H)$ . It is an  *$H$ -factor* if and only if  $\delta_H(F) = 0$ . Denote

$$(2.4) \quad I_H(v) = \{\Phi_F(v) : F \text{ is an } H\text{-optimal subgraph}\}.$$

The vertex set  $V(G)$  can be decomposed as  $V(G) = A_H \sqcup B_H \sqcup C_H \sqcup D_H$ , where

$$C_H = \{v \in V(G) : I_H(v) \subseteq H(v)\},$$

$$A_H = \{v \in V(G) \setminus C_H : \min I_H(v) \geq \max H(v)\},$$

$$B_H = \{v \in V(G) \setminus C_H : \max I_H(v) \leq \min H(v)\}, \quad \text{and}$$

$$D_H = V(G) \setminus A_H \setminus B_H \setminus C_H.$$

The 4-tuple  $(A_H, B_H, C_H, D_H)$  is said to be the  *$H$ -decomposition* of  $G$ . In [7, Corollary (2.4)], Lovász gave the next result.

**Lemma 2.1** (Lovász). *The graph  $G$  has no edge between the vertex subsets  $C_H$  and  $D_H$ .*

Let  $X$  and  $Y$  be disjoint vertex subsets of the graph  $G$ . Denote by  $E(X, Y)$  the set of edges with one end in  $X$  and the other end in  $Y$ . Define the set function  $H_{X,Y}: V(G) - X - Y \rightarrow 2^{\mathbb{Z}}$  by

$$H_{X,Y}(z) = H(z) - \sum_{e: e(Y)-e(X)=1} e(z).$$

In particular, we denote  $H_X = H_{X, \emptyset}$ . On the other hand, define

$$\nu(X, Y) = \sum_{e: e(Y) - e(X) \geq 1} |e(Y) - e(X)| \quad \text{and} \quad \nu(X) = \nu(X, \emptyset).$$

Note that for any edge  $e$ , a vertex in the set  $X$  having a non-trivial contribution to the number  $e(X)$  must be an end of  $e$ . It follows that  $e(X) \in \{0, \pm 1, \pm 2\}$  for any edge  $e$  and for any vertex subset  $X$ . Therefore, we have

$$\begin{aligned} \nu(X) &= \sum_{-e(X) \geq 1} |e(X)| \\ (2.5) \quad &= |\{e \in E(G): e(X) = -1\}| + 2|\{e \in E(G): e(X) = -2\}|. \end{aligned}$$

For any vertex subset  $S$ , we denote by  $G[S]$  the subgraph induced by  $S$ . Denote the number of components of  $G[S]$  by  $c(S)$ . In [7, Theorem (4.3)], Lovász established a formula for the number  $\delta(H)$ .

**Theorem 2.2** (Lovász). *We have*

$$\delta(H) = c(D_H) + \sum_{v \in B_H} \min H(v) - \sum_{v \in A_H} \max H(v) - \nu(A_H, B_H).$$

For any set  $Y$  of integers, we denote its convex hull by  $[Y] = \{y: \min Y \leq y \leq \max Y\}$ . In [7, Theorem (2.1)], Lovász gave the following property for the vertex subset  $D_H$ .

**Lemma 2.3** (Lovász). *Suppose that  $D_H \neq \emptyset$ . Let  $v \in D_H$ . Then the set  $I_H(v)$  is an allowed set. Moreover, we have*

- (i) *if  $\{u \pm 1\} \subseteq I_H(v)$  and  $u \notin I_H(v)$ , then  $u \in H(v)$  and  $u \pm 1 \notin H(v)$ ;*
- (ii) *neither the intersection  $[I_H(v)] \cap H(v)$  nor the difference  $[I_H(v)] \setminus H(v)$  contains a pair of consecutive integers.*

The graph  $G$  is said to be  $H$ -critical if it is connected and  $D_H = V(G)$ . Lovász [7, Lemma (4.1)] showed the following property.

**Lemma 2.4** (Lovász). *If  $G$  is an  $H$ -critical graph, then  $\delta(H) = 1$ .*

In [7, Theorem (4.2)], he also showed that any component of the subgraph  $G[D_H]$  is  $H_{A_H, B_H}$ -critical. In view of Lemma 2.4, this can be stated as follows.

**Lemma 2.5** (Lovász). *Suppose that  $D_H \neq \emptyset$ . Then for any  $H'$ -optimal subgraph  $F$  of any component of the subgraph  $G[D_H]$ , we have  $\delta_{H'}(F) = 1$ , where  $H' = H_{A_H, B_H}$ .*

## 3. MAIN RESULT

This section devotes to the main results of this paper.

Let  $G$  be a graph with two vertex subsets  $S$  and  $T$ . We denote by  $E(S, T)$  the set of edges  $e$  with one end in  $S$  and the other end in  $T$ . When  $T = V(G) \setminus S$ , we use the notation  $\partial(S) = \partial(T) = E(S, T)$ . Let  $f: V(G) \rightarrow \mathbb{Z}^+$  be a function. Define

$$(3.1) \quad \begin{aligned} J_f^*(v) &= J_f(v) \cup \{-1, -3, -5, \dots\} \\ &= \begin{cases} \{\dots, -5, -3, -1, 1, 3, \dots, f(v)\}, & \text{if } f(v) \text{ is odd} \\ \{\dots, -5, -3, -1, 1, 3, \dots, f(v) - 1, f(v)\}, & \text{if } f(v) \text{ is even} \end{cases} \end{aligned}$$

for each vertex  $v$  of  $G$ . From definition, we see that the set function  $J_f^*$  is allowed. Throughout this section, we let  $(A, B, C, D)$  be the  $J_f^*$ -decomposition of  $G$ . Denote  $I = I_{J_f^*}$ ; see Definition (2.4).

Theorems 3.1 and 3.2 treat graphs of even and odd orders respectively.

**Theorem 3.1.** *Let  $G$  be a connected general graph. Let  $f: V(G) \rightarrow \mathbb{Z}^+$  be a function. Then the graph  $G$  satisfies Condition (1.3), i.e.,*

$$o(G - S) \leq f(S) \quad \text{for all vertex subsets } S.$$

*if and only if it contains a colored  $J_f^*$ -factor for any 2-end-coloring.*

*Proof. Necessity.* Suppose that  $G$  contains a vertex subset  $S$  such that

$$(3.2) \quad o(G - S) \geq f(S) + 1.$$

We shall show that there exists a 2-end-coloring for which  $G$  has no  $J_f^*$ -factors. We consider the 2-end-coloring defined by that an end is colored in red if and only if its incident vertex belongs to the set  $S$ . Let  $F$  be a  $J_f^*$ -factor of  $G$ .

Assume that  $E(F) \cap (\partial_F C) = \emptyset$  for some odd component of the subgraph  $G - S$ . By parity argument, the component  $C$  contains a vertex  $v$  having an even degree in  $F$ . From definition, every edge having  $v$  as an end contributes the weight  $-1$  to the colored degree of  $v$ . Thus the colored degree of  $v$  in the factor  $F$  is a negative even integer, contradicting Definition (3.1) of the function  $J_f^*$ .

Otherwise, we have  $E(F) \cap (\partial_F C) \neq \emptyset$  for every odd component  $C$  of  $G - S$ . It follows that  $|\partial_F S| \geq o(G - S) \geq f(S) + 1$  by Ineq. (3.2). Consequently, there exists a vertex  $u \in S$  such that  $|\partial_F(u)| \geq f(u) + 1$ . Since the colored degree of  $u$  in the factor  $F$  is at least  $|\partial_F(u)|$ , we infer that the vertex  $u$  has colored degree larger than  $f(u)$  in  $F$ , contradicting Definition (3.1) of the function  $J_f^*$  again.

**Sufficiency.** By way of contradiction, let  $G$  be a graph satisfying Condition (1.3) without colored  $J_f^*$ -factors. From Definition (2.3), we have

$$(3.3) \quad \delta(J_f^*) > 0.$$

Assume that  $B \neq \emptyset$ . Let  $v_B \in B$ . From definition, we see that

$$\Phi_F(v_B) \leq \max I(v_B) \leq \min J_f^*(v_B)$$

for any  $J_f^*$ -optimal subgraph  $F$ , contradicting the definition of the function  $J_f^*$ . Therefore, we have

$$(3.4) \quad B = \emptyset.$$

As a consequence, Theorem 2.2 and Ineq. (3.3) imply that

$$(3.5) \quad c(D) > \sum_{v_A \in A} \max J_f^*(v_A) + \nu(A) = f(A) + \nu(A).$$

From Eq. (2.5), we can infer that

$$|\{e \in E(A, D) : e(A) = -1\}| \leq |\{e \in E(G) : e(A) = -1\}| \leq \nu(A).$$

Consequently, there are at most  $\nu(A)$  components  $T$  of the subgraph  $G[D]$  such that

$$\{e \in E(A, T) : e(A) = -1\} \neq \emptyset.$$

In other words, the subgraph  $G[D]$  has at least  $c(D) - \nu(A)$  components such that all edges connecting these components with  $A$  have red ends in  $A$ . Together with Ineq. (3.5), we can suppose that the subgraph  $G[D]$  has  $q$  components  $D_1, D_2, \dots, D_q$  with  $q > f(A)$ , such that

$$e(v_A) \geq 0 \quad \text{for each vertex } v_A \in A \text{ and for all edges } e \in \partial(D'),$$

where  $D' = \cup_{i=1}^q D_i$ . In particular, we find  $D' \neq \emptyset$ . Let  $v \in D'$ . From definition, we have

$$(3.6) \quad (J_f^*)_A(v) = J_f^*(v) - \sum_{e: e(A)=-1} e(v) = J_f^*(v).$$

By Lemma 2.5 and Eq. (3.6), we have  $\delta_{J_f^*}(F_i) = 1$  for any  $J_f^*$ -optimal subgraph  $F_i$  of any component  $D_i$ , where  $i \in [q]$ . In particular, there exists a unique vertex  $v_0 \in V(D_i)$  such that

$$(3.7) \quad \Phi_{F_i}(v_0) \not\leq J_f^*(v_0).$$

Since  $v \in D'$ , we have  $v \notin A \cup C$ . From definition, we have

$$(3.8) \quad \min I(v) \leq \max J_f^*(v) - 1 = f(v) - 1.$$

We claim that

$$(3.9) \quad \max I(v) \leq \begin{cases} f(v) - 1, & \text{if } f(v) \text{ is even;} \\ f(v) + 1, & \text{if } f(v) \text{ is odd.} \end{cases}$$

In fact, Ineq. (3.9) can be shown by Lemma 2.3 and Ineq. (3.8). We handle it in two cases according to the parity of  $f(v)$ .

**Case 1.**  $f(v)$  is even.

Assume that  $f(v) \in I(v)$ . From Lemma 2.3 (ii), we infer that  $f(v) - 1 \notin I(v)$ . Then, from Lemma 2.3 (i), we deduce  $f(v) - 2 \notin I(v)$ . In view of Ineq. (3.8), we derive that the set  $I(v)$  is not allowed, contradicting Lemma 2.3. Therefore, we have  $f(v) \notin I(v)$ . Now, assume  $\max I(v) \geq f(v) + 1$ . Since the set  $I(v)$  is allowed, in view of Ineq. (3.8), we infer that  $\{f(v) \pm 1\} \subseteq I(v)$ . From Lemma 2.3 (i), we deduce that  $f(v) - 1 \notin J_f^*(v)$ , a contradiction. Therefore, we have  $\max I(v) \leq f(v) - 1$ .

**Case 2.**  $f(v)$  is odd.

Suppose that there exists  $r > f(v) + 1$  such that  $r \in I(v)$ . Then there exists a  $J_f^*$ -optimal subgraph  $F$  such that  $\Phi_F(v) = r$ . Since  $r \geq 3$ , there exists an edge  $e$  such that  $e(v) > 0$ , that is,  $e(v) \in \{1, 2\}$ . Consider the subgraph  $F - e$ .

If  $e(v) = 2$ , then the edge  $e$  is a loop with two red ends. In this case, the distance

$$\text{dist}_{F-e}(v, J_f^*(v)) = r - 2 - f(v) < \text{dist}_F(v, J_f^*(v)) - 2.$$

In view of Definition (2.2), it follows that  $\delta_{J_f^*}(F - e) < \delta_{J_f^*}(F)$ , contradicting the optimality of  $F$ . Otherwise, we have  $e(v) = 1$ . Then

$$\text{dist}_{F-e}(v, J_f^*(v)) = r - 1 - f(v) = \text{dist}_F(v, J_f^*(v)) - 1.$$

Let  $e = uv$ . Since

$$\text{dist}_{F-e}(u, J_f^*(u)) \leq \text{dist}_F(u, J_f^*(u)) + 1,$$

we infer that  $\delta_{J_f^*}(F - e) \leq \delta_{J_f^*}(F)$  from Definition (2.2). Since  $F$  is optimal, namely  $\delta_{J_f^*}(F - e) \geq \delta_{J_f^*}(F)$ , we deduce that the subgraph  $F - e$  is optimal. From Definition (2.4) of  $I(v)$ , we find  $r - 1 \in I(v)$ . Now, we have  $\{r - 1, r\} \subseteq I(v) \setminus J_f^*(v)$ , contradicting Lemma 2.3 (ii). This proves Ineq. (3.9).

We shall show that the cardinalities  $|D_i|$  are odd.

For any  $J_f$ -optimal subgraph  $F_i$ , we have for  $v \in V(D_i)$ ,

$$(3.10) \quad \Phi_{F_i}(v) \leq \begin{cases} f(v) - 1, & \text{if } f(v) \text{ is even;} \\ f(v) + 1, & \text{if } f(v) \text{ is odd.} \end{cases}$$



Recall from Relation (3.7) that  $\Phi_{F_i}(v) \notin J_f^*(v)$  if and only if  $v = v_0$ . We observe that the colored degree  $\Phi_{F_i}(v)$  is even if and only if  $v = v_0$ . On the other hand, we claim that the colored degree sum

$$\sum_{v \in V(D_i)} \Phi_{F_i}(v) = \sum_{e = vv' \in E(F_i) \text{ is not a loop}} (e(v) + e(v')) + \sum_{e \in E(F_i) \text{ is a loop centered at } v} e(v)$$

is even. In fact, from Definition (2.1), the summand  $e(v) + e(v') \in \{0, \pm 2\}$  when the edge  $e = vv'$  is not a loop, and the summand  $e(v) \in \{\pm 2\}$  when the edge  $e$  is a loop centered at  $v$ . It follows that the component  $D_i$  is of odd order. From Lemma 2.1, Ineq. (3.5), and Eq. (3.4), we derive that

$$(3.11) \quad o(G - A) \geq q > f(A),$$

contradicting Condition (1.3). This completes the proof.  $\square$

It is easy to observe that when the integer  $f(v)$  is odd for each vertex  $v$ , the graph  $G$  contains a  $(1, f)$ -odd-factor if and only if it contains a colored  $J_f^*$ -factor for any 2-end-coloring. In this sense, Theorem 3.1 is a generalization of Theorem 1.2. On the other hand, by coloring all ends in red, the sufficiency part of Theorem 3.1 reduces to Egawa et al.'s Theorem 1.5.

By taking  $S = \emptyset$ , we find Condition (1.3) implies the evenness of the order  $|G|$ . For graphs of odd orders, a slightest remedy to Condition (1.3) might be removing the requirement for empty sets  $S$ .

**Theorem 3.2.** *Let  $G$  be a connected general graph of odd order. Let  $f: V(G) \rightarrow \mathbb{Z}^+$  be a function. Then the graph  $G$  satisfies the Tutte-type condition*

$$(3.12) \quad o(G - S) \leq f(S) \quad \text{for all non-empty vertex subsets } S,$$

*if and only if for every 2-end-coloring, either  $G$  is  $J_f^*$ -critical, or  $G$  contains a colored  $J_f^*$ -factor.*

*Proof. Necessity.* Suppose that  $G$  contains a nonempty vertex subset  $S$  satisfying Ineq. (3.2). Same to the necessity part of the proof of Theorem 3.1, we can show that the graph  $G$  admits a 2-end-coloring for which  $G$  has no  $J_f^*$ -factors. It suffices to show that  $G$  is not  $J_f^*$ -critical for the same coloring. Assuming that the graph  $G$  is  $J_f^*$ -critical, namely,  $D = V(G)$ , we shall show that the empty set  $A \cup C$  contains the nonempty set  $S$ , which is absurd. In other words, we will prove that

$$(3.13) \quad \Phi_F(v) \geq f(v) \quad \text{for any } J_f^*\text{-optimal subgraph } F \text{ and for any vertex } v \in S.$$

Let  $F$  be a  $J_f^*$ -optimal subgraph. From Lemma 2.4, we infer that

$$(3.14) \quad \delta_{J_f^*}(F) = 1 \quad \text{for any } J_f^*\text{-optimal subgraph } F.$$

By parity argument, we have  $E(S, T) \neq \emptyset$  for any odd component  $T$  of the subgraph  $G - S$  such that  $v_0 \notin T$ .

Assume that  $v_0 \in S$ . By Ineq. (3.2), we have

$$\sum_{v \in S} \Phi_F(v) \geq o(G - S) \geq f(S) + 1.$$

If there is a vertex  $v \in S$  and a such that  $\Phi_F(v) < f(v)$ , then either there are two other vertices  $v_1, v_2 \in S$  such that  $\Phi_F(v_i) \geq f(v_i) + 1$  for  $i = 1, 2$ , or there is a vertex  $v_3 \in S$  such that  $\Phi_F(v_3) \geq f(v_3) + 2$ . In either case, we have  $\delta_{J_f^*}(F) \geq 2$  taking account of the contributions of the vertices  $v_i$  to the total distance between the optimal subgraph  $F$  and the function  $J_f^*$ , contradicting Eq. (3.14).

Otherwise, we have  $v_0 \notin S$ . In this case, there are at least  $o(G - S) - 1$  odd components  $T$  of the subgraph  $G - S$  such that  $|E_F(S, T)| \geq 1$ . Therefore, we infer that

$$\sum_{v \in S} \Phi_F(v) \geq o(G - S) - 1 \geq f(S).$$

Since  $v_0 \notin S$ , we have  $\Phi_F(v) \in J_f^*(v)$  for any vertex  $v \in S$ . Hence we deduce that  $\Phi_F(v) = f(v)$  for all vertices  $v \in S$ . This guarantees Ineq. (3.13), and completes the proof of the sufficiency part.

**Sufficiency.** By way of contradiction, let  $G$  be a graph satisfying Condition (3.12) without colored  $J_f^*$ -factors, and is not  $J_f^*$ -critical. Suppose that  $V(G) = C \cup D$ . By Lemma 2.1 and the connectivity of  $G$ , we have  $V(G) = C$  or  $V(G) = D$ . In the former case, the graph  $G$  has a  $J_f^*$ -factor, while in the latter case, the graph  $G$  is  $J_f^*$ -critical. Same to the sufficiency part of the proof of Theorem 3.1, we have  $B = \emptyset$ . Thus we find  $A \neq \emptyset$ . Same to the remaining proof of Theorem 3.1, we have  $o(G - A) > f(A)$ , contradicting Condition (3.12). This completes the proof.  $\square$

As an application of Theorem 3.2, we have the following corollary.

**Corollary 3.3.** *Let  $G$  be a connected general graph of odd order, with a 2-end coloring. Let  $f: V(G) \rightarrow \mathbb{Z}^+$  be a function such that  $f(v)$  is even for each vertex  $v$ . Suppose that Condition (3.12) holds true. If the colored degree of each vertex  $v$  is at least  $f(v)$ , then  $G$  contains a colored  $J_f^*$ -factor.*

*Proof.* By Theorem 3.2, it suffices to show that  $G$  is not  $J_f^*$ -critical.

Suppose that  $G$  is  $J_f^*$ -critical. Let  $F$  be a  $J_f^*$ -optimal subgraph with a maximal edge set. By Lemma 2.4, there is a vertex  $v_0$  such that  $\Phi_F(v_0) \notin J_f^*(v_0)$ .

Assume that  $\Phi_F(v_0) \leq f(v_0) - 2$ . From premise, we see that  $\Phi_G(v_0) \geq f(v_0)$ . Thus there is an edge  $e = v_0 u_0$  such that  $e(v_0) \geq 1$ . If the edge  $e$  is a loop, i.e.,  $u_0 = v_0$ , then

the distance between  $\Phi_{F \cup e}(v_0)$  and the function  $J_f^*(v_0)$  is either equal to or one less than the distance between  $\Phi_F(v_0)$  and  $J_f^*(v_0)$ , namely,

$$\text{dist}_{F \cup e}(v_0, J_f^*(v_0)) \in \{0, 1\}.$$

Since the subgraph  $F$  is optimal, it is impossible that  $\text{dist}_{F \cup e}(v_0, J_f^*(v_0)) = 0$ . In other words, the subgraph  $F \cup e$  must be optimal, contradicting the choice of  $F$ . When  $e$  is not a loop, the colored degree  $\Phi_{F \cup e}(v_0)$  must be in the set  $J_f^*(v_0)$ . Since

$$(3.15) \quad |\text{dist}_{F \cup e}(u_0, J_f^*(u_0)) - \text{dist}_F(u_0, J_f^*(u_0))| \leq 1,$$

we infer that the subgraph  $F \cup e$  must be optimal, the same contradiction.

Below we can suppose that  $\Phi_F(v_0) \geq f(v_0) + 1$ . From definition, we have

$$f(v_0) + 1 \in I(v_0).$$

Since  $\delta(F) = 1$ , we find  $\Phi_F(v_0) = f(v_0) + 1 > 0$ . Pick up an edge  $e \in E(F)$  such that  $e(v_0) \geq 1$ . If  $e$  is a loop, then  $\Phi_{F-e}(v_0) = f(v_0) - 1 \in J_f^*(v_0)$ . Thus the subgraph  $F - e$  is a  $J_f^*$ -factor, a contradiction. Otherwise, we can suppose that  $e = u_0v_0$  with  $u_0 \neq v_0$ . In this case, we have  $\Phi_{F-e}(v_0) = f(v_0) \in J_f^*(v_0)$ . By Ineq. (3.15), we infer that the subgraph  $F - e$  is also  $J_f^*$ -optimal. Thus we have

$$f(v_0) \in I(v_0).$$

Since  $\{f(v_0), f(v_0) - 1\} \subset J_f^*(v_0)$ , by Lemma 2.3 (ii), we infer that

$$f(v_0) - 1 \notin I(v_0).$$

From Lemma 2.3, we also see that the set  $I(v_0)$  is allowed. It follows immediately that  $f(v_0) - 2 \in I(v_0)$ . By Lemma 2.3 (i), we find  $f(v_0) \notin J_f^*(v_0)$ , a contradiction. This completes the proof.  $\square$

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